

## Antisymplectic involutions of holomorphic symplectic manifolds

Arnaud Beauville

## ABSTRACT

Let  $X$  be a holomorphic symplectic manifold, of dimension divisible by 4, and  $\sigma$  an antisymplectic involution of  $X$ . The fixed locus  $F$  of  $\sigma$  is a Lagrangian submanifold of  $X$ ; we show that its  $\hat{A}$ -genus is 1. As an application, we determine all possibilities for the Chern numbers of  $F$  when  $X$  is a deformation of the Hilbert square of a K3 surface.

## Introduction

Let  $X$  be an irreducible holomorphic symplectic manifold admitting an antisymplectic involution  $\sigma$  (that is,  $\sigma$  changes the sign of the symplectic form). The fixed locus  $F$  of  $\sigma$  is a Lagrangian submanifold of  $X$ . The main observation of this note is that *when  $\dim(X)$  is divisible by 4, the  $\hat{A}$ -genus of  $F$  is equal to 1*. Our proof, given in §1, rests on a simple computation based on the holomorphic Lefschetz theorem.

In §2 we apply this result when  $X$  is a symplectic fourfold with  $b_2 = 23$  (this holds when  $X$  is the Hilbert square  $S^{[2]}$  of a K3 surface). We show that there are exactly 11 possibilities for the pair of invariants  $(K_F^2, \chi(\mathcal{O}_F))$  of the surface  $F$ , depending on the number of moduli of  $(X, \sigma)$ . In §3 we illustrate our results on a few examples, in particular the *double EPW-sextics* studied by O'Grady [9], which form the only known family of pairs  $(X, \sigma)$  as above of maximal dimension 20.

1. The  $\hat{A}$ -genus of the fixed manifold.

1.1 Throughout this note we consider an irreducible holomorphic symplectic manifold  $X$  [2]. This means that  $X$  is compact Kähler, simply connected, and admits a symplectic 2-form  $\varphi \in H^0(X, \Omega_X^2)$  which generates the  $\mathbb{C}$ -algebra  $H^0(X, \Omega_X^*)$ . We denote by  $\sigma$  an antisymplectic involution of  $X$  (so that  $\sigma^*\varphi = -\varphi$ ).

**Lemma 1.** *The fixed locus  $F$  of  $\sigma$  is a smooth Lagrangian submanifold of  $X$ .*

*Proof :* Let  $x \in F$ . We have a decomposition  $T_x(X) = T^+ \oplus T^-$  into eigenspaces of  $\sigma'(x)$ . Because of the relation  $\varphi_x(\sigma'(x).u, \sigma'(x).v) = -\varphi_x(u, v)$  for  $u, v \in T_x(X)$ , the two eigenspaces are isotropic, and therefore Lagrangian. Since  $T^+ = T_x(F)$ , the lemma follows. ■

1.2 Observe that the existence of the antisymplectic involution  $\sigma$  forces  $X$  to be *projective*: indeed, let  $H^2(X, \mathbb{Q})^+ \subset H^2(X, \mathbb{R})^+$  be the  $(+1)$ -eigenspaces of  $\sigma^*$  in  $H^2(X, \mathbb{Q}) \subset H^2(X, \mathbb{R})$ . The space  $H^2(X, \mathbb{R})^+$  is contained in  $H^{1,1}$ , and contains a Kähler class; since  $H^2(X, \mathbb{Q})^+$  is dense in  $H^2(X, \mathbb{R})^+$ , it also contains a Kähler class, which is ample.

1.3 The  $\hat{A}$ -genus  $\hat{A}(M)$  of a compact manifold  $M$  is a rational number which can be expressed as a polynomial in the Pontrjagin classes of  $M$  ([7], §26). When  $M$  is a complex manifold of

dimension  $n$ , we have

$$\hat{A}(M) = \int_M \text{Todd}(M) e^{-\frac{c_1(M)}{2}}$$

where  $\int_M : H^*(M, \mathbb{Q}) \rightarrow \mathbb{Q}$  is the evaluation on the fundamental class of  $M$  (see [7], p. 13, formula (12)). If we extend the Euler-Poincaré characteristic  $\chi$  as a  $\mathbb{Q}$ -linear homomorphism  $K(M) \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ , we have  $\hat{A}(M) = \chi(\frac{1}{2}K_M)$ , where  $K_M$  is the canonical bundle of  $M$ .

**Theorem 1.** *Let  $X$  be an irreducible symplectic manifold with  $4 \mid \dim(X)$ ,  $\sigma$  an antisymplectic involution of  $X$ ,  $F$  its fixed manifold. Then  $\hat{A}(F) = 1$ .*

*Proof :* Since  $F$  is Lagrangian (lemma 1), the symplectic form of  $X$  induces an isomorphism  $T_F \xrightarrow{\sim} N_{F/X}^*$ . We apply the holomorphic Lefschetz formula ([1], 4.6):

$$\sum_i (-1)^i \text{Tr } \sigma^*|_{H^i(X, \mathcal{O}_X)} = \int_F \text{Todd}(F)(\text{ch} \wedge N_{F/X}^*)^{-1} = \int_F \text{Todd}(F)(\text{ch} \wedge T_F)^{-1}.$$

Because  $X$  is irreducible symplectic,  $\sigma^*$  acts as  $(-1)^i$  on  $H^{2i}(X, \mathcal{O}_X)$ ; since  $\dim(X)$  is divisible by 4 this implies that the above expression is equal to 1.

As usual we write the Chern polynomial  $c_t(T_F) = \prod_i (1 + t\gamma_i)$ , where the  $\gamma_i$  live in some overring of  $H^*(F)$ . We have

$$\text{Todd}(F) = \prod_i \frac{\gamma_i}{1 - e^{-\gamma_i}} \quad \text{and} \quad \text{ch}(\wedge T_F) = \sum_{i_1 < \dots < i_k} e^{\gamma_{i_1} + \dots + \gamma_{i_k}} = \prod_i (1 + e^{\gamma_i}),$$

hence

$$\text{Todd}(F)(\text{ch} \wedge T_F)^{-1} = 2^{-n} e^{-c_1} \prod_i \frac{2\gamma_i}{1 - e^{-2\gamma_i}}, \quad \text{with } n = \dim(X) \text{ and } c_1 = c_1(T_F).$$

Writing  $\text{Todd}(F) = \sum_k \text{Todd}(F)_k$ , with  $\text{Todd}(F)_k \in H^{2k}(F, \mathbb{Q})$ , we find

$$\int_F \text{Todd}(F)(\text{ch} \wedge T_F)^{-1} = 2^{-n} \sum_k \int_F \frac{(-c_1)^k}{k!} 2^{n-k} \text{Todd}(F)_{n-k} = \int_F \text{Todd}(F) e^{-\frac{c_1}{2}},$$

hence  $\hat{A}(F) = 1$ . ■

Note that the argument applies also when  $\dim(X) \equiv 2 \pmod{4}$  but gives the trivial equality  $\hat{A}(F) = 0$ .

## 2. Symplectic fourfolds

2.1 When  $\dim(X) = 4$  the fixed locus  $F$  is a surface (not necessarily connected). In that case  $\hat{A}(F)$  is equal to  $-\frac{1}{8} \text{sign}(F)$ , where  $\text{sign}(F)$  is the signature of the intersection form on  $H^2(F, \mathbb{R})$  (see [7], 1.5, 1.6 and 8.2.2); we have

$$\text{sign}(F) = \frac{1}{3}(K_F^2 - 2e(F)) = K_F^2 - 8\chi(\mathcal{O}_F),$$

where  $e(F)$  is the topological Euler characteristic of  $F$ , and we put  $K_F^2 = \sum_i K_{F_i}^2$  if  $F_1, \dots, F_p$  are the connected components of  $F$ .

Therefore Theorem 1 gives

$$\text{sign}(F) = K_F^2 - 8\chi(\mathcal{O}_F) = -8 \quad \text{and} \quad K_F^2 - 2e(F) = -24.$$

We will be able to say more when the action of  $\sigma$  on  $H^2(X)$  controls the action on  $H^4(X)$ , that is, when the canonical map  $\text{Sym}^2 H^2(X) \rightarrow H^4(X)$  is an isomorphism. By [6] this happens if

and only if  $b_2(X) = 23$ . This is the case for one of the two families of symplectic fourfolds known so far, namely the family of Hilbert schemes  $S^{[2]}$  of a K3 surface  $S$  (and their deformations).

**Theorem 2.** *Let  $X$  be a symplectic fourfold with  $b_2(X) = 23$ ,  $\sigma$  an antisymplectic involution of  $X$ ,  $F$  its fixed surface. Let  $t$  denote the trace of  $\sigma^*$  acting on  $H^{1,1}(X)$ .*

a) *We have*

$$K_F^2 = t^2 - 1 \quad \chi(\mathcal{O}_F) = \frac{1}{8}(t^2 + 7) \quad e(F) = \frac{1}{2}(t^2 + 23)$$

b) *The local deformation space of  $(X, \sigma)$  is smooth of dimension  $\frac{1}{2}(21 - t)$ .*

c) *The integer  $t$  can take any odd value with  $-19 \leq t \leq 21$ .*

*Proof :* The classical Lefschetz formula reads

$$e(F) = \sum_i (-1)^i \operatorname{Tr} \sigma^*_{|H^i(X)}$$

where we put  $H^*(X) := H^*(X, \mathbb{Q})$ . In the case  $b_2 = 23$ , the odd degree cohomology vanishes, and the natural map  $\operatorname{Sym}^2 H^2(X) \rightarrow H^4(X)$  is an isomorphism [6]. Let  $a$  and  $b$  be the dimensions of the  $(+1)$ - and  $(-1)$ -eigenspaces of  $\sigma^*$  on  $H^2(X)$ . We have  $a + b = 23$  and  $a - b = t - 2$ . Then

$$\operatorname{Tr} \sigma^*_{|H^4(X)} = \frac{1}{2}a(a+1) + \frac{1}{2}b(b+1) - ab = \frac{1}{2}(t-2)^2 + \frac{23}{2},$$

$$\text{and } e(F) = 2 + 2 \operatorname{Tr} \sigma^*_{|H^2(X)} + \operatorname{Tr} \sigma^*_{|H^4(X)} = 2 + 2(t-2) + \frac{1}{2}(t-2)^2 + \frac{23}{2} = \frac{1}{2}(t^2 + 23);$$

using (2.1) we deduce the other formulas of a).

We have  $H^2(X, T_X) \cong H^2(X, \Omega_X^1) = 0$ , hence the versal deformation space  $\operatorname{Def}_X$  of  $X$  is smooth and can be locally identified with  $H^1(X, T_X)$ ; the involution  $\sigma$  gives rise to an involution of  $\operatorname{Def}_X$ , which under the above identification corresponds to  $\sigma^*$  acting on  $H^1(X, T_X)$ . Thus the deformation space of  $(X, \sigma)$  is identified with the  $(+1)$ -eigenspace of  $\sigma^*$ . Since  $\sigma^*\varphi = -\varphi$ , this eigenspace is mapped by the isomorphism

$$H^1(X, T_X) \xrightarrow{i(\varphi)} H^1(X, \Omega_X^1)$$

to the  $(-1)$ -eigenspace of  $\sigma^*$  in  $H^1(X, \Omega_X^1)$ . With the previous notation the dimension of this eigenspace is  $b - 2 = \frac{1}{2}(21 - t)$ , which proves b).

Let us prove c). Since  $\sigma$  preserves some Kähler class we have  $a = \frac{1}{2}(t + 21) \geq 1$ , hence  $t \geq -19$ ; since  $\sigma^*\varphi = -\varphi$  we have  $b = \frac{1}{2}(25 - t) \geq 2$ , hence  $t \leq 21$ . We will construct in 3.2, 3.3 and 3.4 below examples with all possible values of  $t$ . ■

**Corollary.** *The pair  $(K_F^2, \chi(\mathcal{O}_F))$  can take any of the values  $(0, 1)$ ,  $(8, 2)$ ,  $(24, 4)$ ,  $(48, 7)$ ,  $(80, 11)$ ,  $(120, 16)$ ,  $(168, 22)$ ,  $(224, 29)$ ,  $(288, 37)$ ,  $(360, 46)$ ,  $(440, 56)$ .*

### 3. Examples

**3.1** Let  $S$  be a K3 surface,  $\sigma$  an antisymplectic involution of  $S$ ; it extends to an antisymplectic involution  $\sigma^{[2]}$  of the Hilbert scheme  $X = S^{[2]}$ , which preserves the exceptional divisor  $E$  (the locus of non-reduced subschemes). We have  $H^{1,1}(X) = H^{1,1}(S) \oplus \mathbb{C}[E]$ , hence  $t = \operatorname{Tr} \sigma^*_{|H^{1,1}(S)} + 1$ . The fixed locus of  $\sigma$  is a curve  $\Gamma$  on  $S$  (not necessarily connected); the Lefschetz formula for  $\sigma$  gives  $t = e(\Gamma) + 1$ . The list of all possibilities for  $\Gamma$  can be found in [8].

The fixed surface  $F$  of  $\sigma^{[2]}$  is the union of the symmetric square  $\Gamma^{(2)}$  and the quotient surface  $S/\sigma$ .

3.2 Let  $C$  be an irreducible plane curve of degree 6, with  $s$  ordinary double points ( $0 \leq s \leq 10$ ) and no other singularities. Let  $\pi : S' \rightarrow \mathbb{P}^2$  be the double covering of  $\mathbb{P}^2$  branched along  $C$ ,  $S$  the minimal resolution of  $S'$ , and  $\sigma$  the involution of  $S$  which exchanges the sheets of  $\pi$ . The fixed locus  $\Gamma$  of  $\sigma$  is the normalization of  $C$ ; thus  $e(\Gamma) = -18 + 2s$ , and  $t = -17 + 2s$ .

3.3 For each integer  $r$  with  $1 \leq r \leq 10$ , there exists a K3 surface  $S$  and an involution of  $S$  whose fixed locus is the disjoint union of  $r$  rational curves [8]. Then  $e(\Gamma) = 2r$  and  $t = 2r + 1$ . Together with the previous example this gives all integers  $t$  appearing in Theorem 2 c), except  $t = -19$ .

3.4 The case  $t = -19$  is particularly interesting, because when it holds the deformation space of  $(X, \sigma)$  has maximal dimension 20 (Theorem 2, b)). The space  $H^2(X, \mathbb{Q})^+$  is one-dimensional, generated by an ample class (1.2); the deformation space of  $(X, \sigma)$  coincides locally with the deformation space of  $X$  as a polarized variety. We know only one example of this situation: O'Grady has constructed a 20-dimensional family of projective symplectic fourfolds which are double coverings of certain sextic hypersurfaces in  $\mathbb{P}^5$ , called EPW-sextics [9]. The corresponding involution is antisymplectic and must satisfy  $t = -19$  by Theorem 2 b). The fixed surface  $F$  is connected, and from Theorem 2 a) we recover the invariants  $K_F^2 = 360$ ,  $\chi(\mathcal{O}_F) = 46$  already obtained in [10].

3.5 As explained in [5]<sup>†</sup>, the above pairs  $(X, \sigma)$  specialize to  $(S^{[2]}, \tau)$ , where  $S$  is a smooth quartic surface in  $\mathbb{P}^3$  which contains no line, and  $\tau$  associates to a length 2 subscheme  $z \in S^{[2]}$  the residual subscheme in the intersection of  $S$  and the line spanned by  $z$ . The fixed locus becomes the surface  $B$  of bitangents to  $S$ ; this explains why  $B$  has the same invariants  $K_B^2 = 360$ ,  $\chi(\mathcal{O}_B) = 46$ , as already observed by Welters [11].

3.6 There are many other examples, which give rise to interesting exercises. Here is one: we start with the involution  $\iota$  of  $\mathbb{P}^5$  given by  $\iota(X_0, \dots, X_5) = (-X_0, X_1, \dots, X_5)$ . Let  $V \subset \mathbb{P}^5$  be a smooth cubic threefold invariant under  $\iota$ : its equation must be of the form  $X_0^2 L(X_1, \dots, X_5) + G(X_1, \dots, X_5) = 0$ , where  $L$  is linear and  $G$  cubic. The Fano variety  $X$  of lines contained in  $V$  is a symplectic fourfold [3], and  $\iota$  defines an involution  $\sigma$  of  $X$ .

The fixed points of  $\iota$  in  $\mathbb{P}^5$  are  $p = (1, 0, \dots, 0)$  and the hyperplane  $H$  given by  $X_0 = 0$ . A line  $\ell \in X$  is preserved by  $\iota$  if and only if it contains at least two fixed points; this means that either  $\ell$  contains  $p$ , or it is contained in  $H$ . The lines passing through  $p$  are parametrized by the cubic surface  $S \subset H$  given by  $L = G = 0$ ; the lines contained in  $H$  form the Fano surface  $T$  of the cubic threefold  $G = 0$  in  $H$ . Thus the fixed surface  $F$  of  $\sigma$  is the disjoint union of  $S$  and  $T$ .

Using the canonical isomorphism  $H^{1,1}(X) \xrightarrow{\sim} H^{2,2}(V)$  [3] and Griffiths' description of the cohomology of the hypersurface  $V$ , one finds easily  $t = -7$ . Then Theorem 2 a) gives  $K_F^2 = 48$  and  $\chi(\mathcal{O}_F) = 7$ . Since  $K_S^2 = 3$  and  $\chi(\mathcal{O}_S) = 1$ , we recover the values  $K_T^2 = 45$  and  $\chi(\mathcal{O}_T) = 6$  [4].

## References

1. M. ATIYAH and I. SINGER, 'The index of elliptic operators. III', *Ann. of Math.* (2) **87** (1968), 546–604.
2. A. BEAUVILLE, 'Variétés kählériennes dont la première classe de Chern est nulle', *J. of Diff. Geometry* **18** (1983), 755–782.
3. A. BEAUVILLE and R. DONAGI, 'La variété des droites d'une hypersurface cubique de dimension 4', *C. R. Acad. Sci. Paris Sér. I Math.* **301** (1985), no. 14, 703–706.

---

<sup>†</sup> I am indebted to K. O'Grady for pointing out this paper to me, thus correcting an inaccurate remark in the first version of this note.

4. H. CLEMENS and P. GRIFFITHS, 'The intermediate Jacobian of the cubic threefold', *Ann. of Math.* (2) **95** (1972), 281–356.
5. A. FERRETTI, 'The Chow ring of double EPW sextics', *Preprint* arXiv:0907.5381.
6. D. GUAN, 'On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four', *Math. Res. Lett.* **8** (2001), no. 5-6, 663–669.
7. F. HIRZEBRUCH, 'Topological methods in algebraic geometry', *Classics in Mathematics*. Springer-Verlag, Berlin, 1995.
8. V. NIKULIN, 'On the quotient groups of the automorphism groups of hyperbolic forms by the subgroups generated by 2-reflections. Algebro-geometric applications', *J. Soviet Math.* **22** (1983), 1401–1476.
9. K. O'GRADY, 'Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics', *Duke Math. J.* **134** (2006), no. 1, 99–137.
10. K. O'GRADY, 'Irreducible symplectic 4-folds numerically equivalent to  $(K3)^{[2]}$ ', *Commun. Contemp. Math.* **10** (2008), no. 4, 553–608.
11. G. WELTERS, 'Abel-Jacobi isogenies for certain types of Fano threefolds', *Mathematical Centre Tracts* **141**. Mathematisch Centrum, Amsterdam, 1981.

Arnaud Beauville  
Laboratoire J.-A. Dieudonné  
UMR 6621 du CNRS  
Université de Nice  
Parc Valrose  
F-06108 Nice cedex 2, France  
arnaud.beauville@unice.fr